TEACHING CHILDREN TO REPRESENT RECTANGULAR ARRAYS

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The development of young children's representations of a rectangular array as [perpendicular] intersecting sets of parallel lines is the focus of this paper. Drawing an array using lines, rather than drawing each rectangular area unit separately should assist children to represent the correct numerical structure and to perceive a row as a single, composite unit. Without a knowledge of the numerical structure of an array, it is unlikely that children will be able to apply repeated addition or multiplication skills to determine the number of elements in an array.

To draw a rectangular array, it seems necessary for children to grasp three properties of an array: that the area units must be congruent, that the units are aligned, and that each row has the same number of units. A teaching experiment was undertaken to determine the effect of stressing these features on children's array drawings. Emphasising that the units should be the same size did not help the children to draw the correct array structure, while stressing the alignment of units or that there is an equal number of units in each row, did seem to assist children to draw an array as lines.

Area is a particularly important topic in school mathematics. It is one of the most commonly used domains of measure in everyday life, and it is the basis for many models used by teachers and textbooks to explain numbers and operations with numbers (Hirstein, Lamb, & Osborne, 1978). This paper reports on one aspect of a study of the effects of children's perceptions of arrays on the development of their concepts of rectangular area measurement. The teaching experiment that is described in this paper arose from an analysis of the responses to a sequence of array drawing tasks given to 140 children in Years 1 to 4 of primary school.

Knowledge of array multiplication is a key prerequisite to understanding the formula for the area of a rectangle. An essential step in applying multiplication to arrays would seem to be a realisation that individual units in a row [column] can be grouped to form a new composite unit. To consider a row [column] as a single unit, however, is a fundamental change in a child's conception of what constitutes a unit. The change from children's constructions of single entities as units to their constructions of composite or multiple entities as units has been explored in depth by Steffe (1988; 1992) who believes that such a change is crucial to the development of multiplication. Hiebert (1988) noted that this change is not trivial because "a change in the nature of the unit is a change in the most basic entity of arithmetic" (p. 2).

Although arrays are an inherent part of multiplication, Outhred and Mitchelmore (1991) found that basic understandings of arrays were not well developed in children in Years 1 and 2 and some children in Years 3 and 4 seemed to have a limited knowledge of the essential features of array structure. Other researchers have reported that children may not identify array models with multiplication (Beattys and Mayer, 1989; English, 1982)
One aspect of children's knowledge of array structure would seem to be a perception of the array as groups of rows [or columns] that can represented by lines rather than as a collection of individual units. The use of lines may assist children to construct composite units as the lines emphasise the action of grouping individual units into a composite unit.

THE TEACHING EXPERIMENT

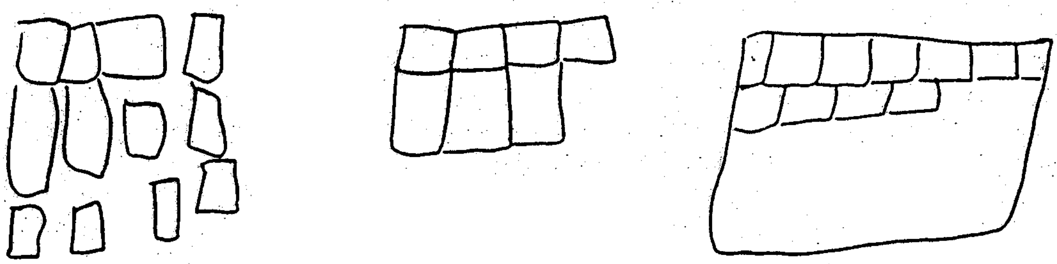
The results of the main study interviews indicated that children need to grasp three basic properties to draw an array. These properties are:

1. that the units must be congruent
2. that the units must be aligned both vertically and horizontally
3. that each row [column] has an equal number of units.

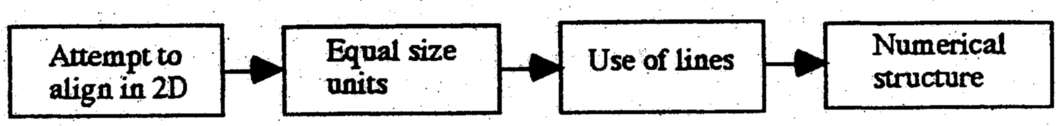
The first two properties are referred to as the spatial structure of an array, and the third property as the numerical structure. The aim of the study reported in this paper was to evaluate which of three instructional methods was most effective in teaching children to draw arrays using lines. Each method emphasised one of the above properties of an array and only square units were considered. The three properties seemed to be independent as an analysis of the interview data provided examples of children's drawings in which each property occurred in isolation (see Figure 1). The properties were also found to occur in combinations of two categories without the third. For example, the squares might have been drawn so that they were aligned vertically and horizontally and they were approximately the same size, but each row might not have the same number of squares.

If rows of congruent units [squares] are drawn without gaps, it follows that the columns so formed must be aligned, whereas vertical and horizontal alignment of squares does not necessarily imply congruence. Thus drawing children's attention to the size of the squares should correct any alignment errors and ensure that the numerical structure is correct, whereas alerting children to the alignment of the squares might not result in congruent squares being drawn. Similarly, pointing out to children that there is an equal number of squares in each row [column] does not subsume the properties of alignment and congruence.

Figure 1 Examples of the three properties of arrays occurring independently of each other



Therefore, the hypothesis to be tested is that for children who have begun to connect units in two dimensions (2D), the three teaching approaches will be ordered from most (1) to least effective (3); that is, the hypothesised development of the array structure is:



This hypothesis assumes that in general, children have to master the geometrical structure of the array before the numerical structure. This assumption may not be warranted when the array is small or when a concrete model is present for children to copy.

SELECTION OF THE SAMPLE

A group pre-test was used to select children who were at a particular level in their development of the array concept. The pre-test was given to two class groups of children in each of Years 1 and 2 in two schools; it consisted of two tasks. The first task was to make a 3x4 array using 4 cm cardboard squares and then to draw the array. In the second task (an indicated grid), marks were shown on the sides of a 5 cm x 8 cm rectangle to indicate the number of 1 cm squares that would fit along each side; the child was asked to draw the squares to cover the rectangle. A 1 cm square was shown next to the rectangle to indicate the size of the squares.

The model task was a good introductory activity to explain the concept of covering to the children. A large rectangle on a sheet of blue cardboard was covered with stick-on yellow squares while the children watched. Then the cardboard was turned over to show the completed model with the squares taped down to ensure that they were carefully aligned. The children were asked to draw the squares so that their drawings looked exactly the same as the model. Next the children were asked to complete the indicated grid task. An indicated grid drawn on a large sheet of cardboard was used to illustrate the instructions to the children.

Children who attempted to align the squares in two dimensions but who had not yet mastered this skill were chosen to be part of the sample. These children were allocated to the three groups so that the groups were balanced for year level, gender and for similarities among the children's pre-test drawings on the indicated grid task. Two of the groups consisted of 15 children, the other of 16 children.

THE TEACHING SESSION

One individual teaching session was given to each child and the indicated grid task was presented as a post-test at the end of the session. The same procedure was followed for each teaching method. The first teaching activity was a variation of the model task; the child made an array by covering a rectangle with cardboard squares. A transparent overlay showing the array drawn as two sets of lines was placed over the squares to emphasise the array structure; the child's attention was drawn to the constant size of the squares, to their alignment, or to the number of squares in each row [column] depending on the instructional group to which the child belonged. This initial activity was repeated with different rectangles and squares of different sizes. Next the child was shown a set of pictures of arrays. There were three different sets of pictures and in each set, a single feature was stressed. The pictures exemplified some of the different types of children's drawings (see below). The child was asked to suggest how the drawings could be improved. If the child did not mention the specific property, the teacher gave a non-committal response, while comments that concerned the relevant feature were discussed. After this task the child made a large array (5 x 6) that s/he was asked to copy. A large array was used to elicit drawing errors. The differences between each child's drawing and the model were pointed out, again stressing the appropriate feature. A similar procedure was followed for a partially completed array (squares were shown on two adjacent sides of a 6 x 7 rectangle); the child was asked to complete the drawing. Any difficulties that involved the relevant property were then discussed. Three numerical tasks were given after the post-test to see if an emphasis on the numerical structure of arrays assisted children to use the row or columns structure to systematise their counting.

RESULTS

The results of the earlier study (Outhred & Mitchelmore, 1991) found that children's drawings of arrays could be classified on two different (but related) dimensions, spatial and numerical structure. The spatial categories that were used were:

1 the child did not attempt to align the squares in two dimensions;
2 the child attempted to connect the squares (drawn individually) in two dimensions but experienced difficulties with alignment;
3 individual squares were drawn, that were reasonably well aligned;
4 the child made some use of lines in drawing the array. Several alternative methods were observed: lines might be drawn in one dimension only (that is, either horizontally or vertically, but not both); some squares might be drawn individually and the remainder of the array drawn as horizontal and vertical lines; or the array might be drawn as two (perpendicular) sets of parallel lines.

The numerical categories were:

1 there were not an equal number of elements in each row or column;
2 equal numbers of elements were drawn in each row [column] but the number of elements was incorrect (i.e., the array was not 5x8);

3 the array was numerically correct.

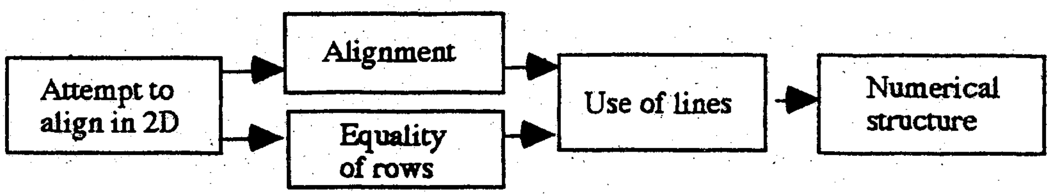
The children were selected for the sample because they had been given a spatial score of 2 and a numerical score of 1 for the indicated-grid pre-test task. The spatial and numerical category scores were used to analyse the differences among the three groups. The post-test spatial and numerical category scores on the indicated grid task are given in Table 1. A Kruskal-Wallis non-parametric test indicated that there are significant differences among the three groups ($p=.009$) for the spatial category scores but not for the numerical category scores ($p=.09$).

Table 1 Frequency distribution of the spatial and numeric category scores for the indicated grid post-test task.

	Spatial category			Numerical category		
Teaching group	2	3	4	1	2	3
Size	11	3	1	10	1	4
Alignment	4	2	10	4	2	10
Equality of rows	8	1	6	7	1	7

It can be seen from Table 1 that the hypothesised model for the development of the array structure is not supported as teaching children that the squares must be equal in size was the least effective strategy. Teaching children that "the squares must be underneath each other so that the edges meet each other" or emphasising that the number of squares in each row must be the same seemed to be similarly effective teaching strategies to assist children to develop the strategy of representing squares by drawing lines. Why might this be so? It was not the result that was expected. A possible explanation might be that size is too global a description to help the children draw congruent squares. To use information about size children must derive the essential drawing strategy, that is, squares that are equal in size will necessarily be aligned. In contrast, the teaching procedures for both alignment and number emphasised that the squares should be drawn underneath each other. While this was made explicit when stressing alignment, it was implicit in the idea of an equal number in each row, as children are first taught to make equal groups by one-to-one matching.

Does the evidence from the teaching experiment support the conjecture in the hypothesised model that it is the development of lines that leads to a numerically correct structure? The data from the pre- and post test tasks did seem to support this conjecture. It is not imperative that children use lines before they draw the correct numerical structure for the indicated grid task, but only four of the 21 children in numerical category 3 constructed an array without drawing lines. While these four children drew the correct numerical structure by drawing arrays of individual squares, usually this strategy resulted in alignment problems and thus an incorrect numerical structure. The results in Table 1 indicate that the model for the development of array structure for the indicated grid task should be represented as:



Correct numerical structures were achieved quite differently for the two tasks that were given in the pre-test. When copying from a model, drawing individual squares was the most commonly used and successful strategy as children found it difficult to draw an array of lines when the unit subdivisions were not shown. The children who attempted to draw lines usually drew equal rows but they had difficulty in subdividing the rectangle into the correct number of squares. For the indicated grid task an individual-square strategy was rarely successful, perhaps because the array to be drawn was larger, the size of the squares was constrained by size of the rectangle, and the children had to visualise the array structure.

WHAT DO CHILDREN NOTICE ABOUT INCORRECT DRAWINGS

That's kind of nearly, just like that, but it's got little gaps there, it would be if you had joined those up. This one is a little bit too big, they've all got to be the same size. It's easy for me.

The responses to the task in which children were asked to comment on other children's drawings were examined in more detail by analysing the audio tapes of the teaching sessions. This was done to determine the features that children commonly noticed, and whether children mentioned the property that was emphasised in the teaching session. Children's comments about the three arrays were classified into four categories, size, alignment, number and gaps. The 'gaps' category was included as children frequently commented on spaces between the squares.

The feature that most children noticed was the comparative sizes of the squares, whether this feature had been stressed in the teaching session or not. The presence of gaps was also commonly noticed by most children, except for children in the group in which size had been emphasised. That the squares were aligned and that the number of squares in each row [column] was the same were only commented on by children who had been taught about these features. The pre-occupation with size might account for the frequency with which children added extra rows, columns or squares to their drawings. The urge to make the "squares" look the same size seemed to over-ride any other considerations. Thus children who had tried to align rows with each other but who could not keep the columns vertical (the squares in the bottom rows were smaller) would add a few extra squares to the last few rows because "it [the drawing] doesn't look right".

DISCUSSION

The results of this experiment indicated that teaching children that the squares in an array were all the same size was not the most effective method to help the children to perceive that rows and columns could be represented by lines and to draw a numerically correct array. This result was not expected because the property that the squares are the same size logically subsumed the other two properties of an array, alignment of the squares and the equality of rows [columns]. Teaching children that the squares are aligned or that there are the same number of squares in each row [column], however, both seemed to be effective methods of moving children from drawing squares individually to representing groups of squares using lines.

Children's responses to the task in which they were asked to comment on other children's drawings suggested that the two key features that assisted children to draw lines were not noticed by children unless these features had been specifically taught. The need to point out the alignment of the squares and the numerical structure of arrays to children may explain the effectiveness of emphasising these two properties in helping children to realise that rows and columns could be represented by lines. These two methods also gave more specific guidelines for drawing an array accurately as individual squares than teaching children that the squares should be the same size. An accurate representation of an array as individual squares seemed to be an important step in shifting to a line strategy.

Therefore, although some drawing techniques must be developed, the most important skill in representing an array seems to be based on an understanding of a fundamental property of rectangular arrays in general: the elements of an array are collinear in two directions. It is this property that allows the partitioning of the array into equal rows and columns which may be a first step in perceiving that these groups of single units may be reconstituted as composite units.

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